ON THE BEST APPROXIMATION OF THE DIFFERENTIATION OPERATOR[1]

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Abstract: In this paper we give a solution of the problem of the best approximation in the uniform norm of the differentiation operator of order k by bounded linear operators in the class of functions with the property that the Fourier transforms of their derivatives of order n $(0 < k < n)$ are finite measures. We also determine the exact value of the best constant in the corresponding inequality for derivatives.

Key words: Differentiation operator, Stechkin's problem, Kolmogorov inequality

This paper is devoted to studying the best approximation in the uniform norm on the real line of the differentiation operator of order k by bounded linear operators in the class of functions with the property that the Fourier transforms of their derivatives of order n $(0 < k < n)$ are finite measures. S. B. Stechkin [8] was the first who studied the problem of the best approximation of the differentiation operator (or, more generally, of an unbounded operator) by bounded ones. In particular, he noticed that this problem is connected to the best constant in an inequality between the norms of the derivatives. Later these questions were studied by Yu. N. Subbotin, L. V. Taikov, V. N. Gabushin, A. P. Buslaev, the author, and others (see [1–6, 8–10] and the bibliography therein).

Let $C = C(-\infty, \infty)$ be the space of continuous bounded (complex-valued) functions on the real line with the uniform norm, let M be the space of finite (complex) Borel measures on $(-\infty, \infty)$ with the norm equal to the total variation $\bigvee \mu$ of a measure μ, and let $L_r,\ 1 \le r < \infty$, be the space of measurable functions with the (finite) norm

$$\|x\|_r = \left(\int |x(t)|^r \, dt \right)^{\frac{1}{r}}.$$

The Fourier transform $\widetilde{x}$ of a function $x \in L_1$ is defined by the formula

$$\widetilde{x}(t) = \int e^{-2\pi t\eta i}\, x(\eta)\, d\eta.$$

In this case the inverse Fourier transform has the form

$$\hat{x}(t) = \int e^{2\pi t\eta i}\, x(\eta)\, d\eta.$$

Further on, let S be the space of infinitely differentiable, rapidly decreasing functions on the real line, and let S' be the corresponding dual space of generalized functions. We will denote the value of a functional $\theta \in S'$ on the function $x \in S$ by $\langle \theta, x \rangle$. The Fourier transform $\widetilde{\theta}$ of a functional $\theta \in S'$ is the functional $\widetilde{\theta} \in S$ acting according to the rule $\langle \widetilde{\theta}, x \rangle = \langle \theta, \widetilde{x} \rangle$. If $\theta \in L_\gamma$, $1 \le \gamma \le 2$, then $\widetilde{\theta} \in L_{\gamma'}$, $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$.

[1]The paper was originally published in a hard accessible collection of articles *Approximation of Functions by Polynomials and Splines* (UNTs AN SSSR, Sverdlovsk, 1985), p. 3–14 (in Russian).

Denote by F_n, $n \geq 1$, the set of functions $x \in C$ whose derivatives $x^{(n)}$ of order n are continuous functions such that their Fourier transforms are measures, i. e.

$$x^{(n)}(t) = \int e^{2\pi t\eta i}\, d\mu(\eta), \tag{1}$$

where $\mu = \mu_x = \widetilde{x^{(n)}} \in M$. We will denote the total variation $\bigvee \mu$ of a measure μ in (1) by $\|x^{(n)}\|_V$. We will consider the subclass $\widetilde{Q}_n = \{x \in F_n : \|x^{(n)}\|_V \leq 1\}$ in F_n. We study the problem of the best approximation of the differentiation operator of order k $(0 < k < n)$ on the class $\widetilde{Q}_n$ by the set $\mathcal{L}(N)$ of linear bounded operators T in the space C with the norm $\|T\| = \|T\|_{C\to C} \leq N$. In other words, we study the quantity

$$e(N) = e_{k,n}(N) = \inf\{u(T) : T \in \mathcal{L}(N)\}, \tag{2}$$

where

$$u(T) = u_{k,n}(T) = \sup\{\|x^{(k)} - Tx\|_C : x \in \widetilde{Q}_n\}. \tag{3}$$

Our main results are the following two statements.

Theorem 1. *For each $h > 0$ we have*

$$e_{k,n}(N_{k,n}(h)) = \frac{k}{n} h^{n-k}, \tag{4}$$

where

$$N_{1,2}(h) = \frac{16}{h\pi^3} \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^3}, \tag{5}$$

$$N_{k,n}(h) = \frac{n-k}{n} h^{-k}, \quad n \geq 3, \quad 1 \leq k \leq n-1. \tag{6}$$

Theorem 2. *Functions of the class F_n satisfy the sharp inequality*

$$\|x^{(k)}\|_C \leq K_{k,n}\, \|x\|_C^{\frac{n-k}{n}}\, \|x^{(n)}\|_V^{\frac{k}{n}}, \tag{7}$$

and the smallest possible constant in this inequality is

$$K_{1,2} = \left(\frac{32}{\pi^3} \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^3}\right)^{\frac{1}{2}} > 1,$$

$$K_{k,n} = 1, \quad n \geq 3, \quad 1 \leq k < n. \tag{8}$$

The fact that functions from the set F_n satisfy inequality (7) with some finite constant follows from a result of A. N. Kolmogorov [7], for $\|x^{(n)}\|_C \leq \|x^{(n)}\|_V$. However, one cannot obtain the smallest possible constant in (7) using this approach.

P r o o f of the both theorems will be done simultaneously following the scheme which was developed by S. B. Stechkin [8] and later used by other authors (see, e. g., [1, 4, 5, 9, 10]). Consider

$$\omega(\delta) = \sup\{\|x^{(k)}\|_C : x \in \widetilde{Q}_n,\ \|x\|_C \leq \delta\}, \quad \delta > 0. \tag{9}$$

It follows from the homogenity of $\omega(\delta)$ (see, e.g. [11, p. 116]) that

$$\omega(\delta) = K\,\delta^{\alpha}, \quad \alpha = \frac{n-k}{n}, \tag{10}$$

with $K = \omega(1)$. This fact implies inequality (7), and the smallest possible constant in (7) is $K = K_{k,n} = \omega(1)$. Using S. B. Stechkin's method [8], one can show that $e(N) \geq \omega(\delta) - N\delta = K\delta^{\alpha} - N\delta$

for each $N > 0$ and $\delta > 0$, that is, $K \le N\delta^{1-\alpha} + e(N)\delta^{-\alpha}$. Minimizing the latter expression with respect to $\delta > 0$, we obtain the inequality

$$K^n \le n^n \left(\frac{N}{n-k}\right)^{n-k} \left(\frac{e(N)}{k}\right)^k . \tag{11}$$

Consequently, an upper estimate for $e(N)$ (a concrete operator) gives an upper estimate for K, and a lower estimate for K (a concrete function $x \in F_n$) gives a lower estimate for $e(N)$.

We start the concrete realization of this scheme by considering the case $n = 2$, $k = 1$. First we obtain an upper bound for $e(N)$ using a concrete operator. Let η be an odd 2π-periodic function which is defined on $[0, \pi]$ by the formula $\eta(t) = t - \frac{1}{\pi}t^2$. We have

$$\eta(t) = \sum_{\ell=0}^{\infty} c_\ell \sin(2\ell+1)t, \quad c_\ell = \frac{8}{\pi^2}\frac{1}{(2\ell+1)^3}. \tag{12}$$

It is not difficult to see that the operator $T = T_{1,2}$ defined by the formula

$$(T_{1,2}x)(t) = \frac{1}{2\nu(h)} \sum_{\ell=0}^{\infty} c_\ell \{x(t + (2\ell+1)\nu(h)) - x(t - (2\ell+1)\nu(h))\}, \tag{13}$$

where $\nu = \nu(h) = \frac{\pi h}{2}$, is a linear bounded operator in C and

$$\|T\|_{C\to C} = \frac{1}{\nu}\sum_{\ell=0}^{\infty} c_\ell = \frac{16}{\pi^3 h}\sum_{\ell=0}^{\infty}\frac{1}{(2\ell+1)^3} = N_{1,2}(h). \tag{14}$$

Introduce the function $\varphi(t) = (t - \eta(t))t^{-2}$. To determine its norm in the space C, we notice that $t \ge \eta(t) \ge t - \frac{1}{\pi}t^2$ for all $t \ge 0$, and thus $|\varphi(t)| \le \frac{1}{\pi}$. Furthermore, if $t \in [0, \pi]$, then $\varphi(t) = \frac{1}{\pi}$. Consequently,

$$\|\varphi\|_{C(-\infty,\infty)} = \frac{1}{\pi}. \tag{15}$$

Now let us prove that the representation

$$x'(t) - (T_{1,2}x)(t) = -i\nu(h) \int e^{2\pi t\tau i}\, \varphi(2\pi\tau\nu(h))\, d\mu_x(\tau) \tag{16}$$

holds for functions $x \in F_2$, where $\mu = \mu_x = \widetilde{x''}$ is the measure from representation (1). First assume that a function x and its derivative x'' both belong to L_2. In this case, the function $y = x' - Tx$ belongs to L_2 as well, and it is easy to see that the Fourier transform of the function y has the form $\widetilde{y}(t) = -i\nu\, \varphi(2\pi t\nu)\, \widetilde{x''}(t)$. Taking the inverse Fourier transform, we obtain the expression

$$y(t) = x'(t) - (Tx)(t) = -i\nu \int \varphi(2\pi\tau\nu)\, \widetilde{x''}(\tau)\, e^{2\pi it\tau}\, d\tau, \tag{17}$$

which is representation (16) in this particular case.

Now let x be an arbitrary function from the class F_2. Introduce the functions

$$\zeta(t) = (1+t^2)^{-1}, \quad \zeta_\varepsilon(t) = \zeta(\varepsilon t), \quad z = z_\varepsilon = x\zeta_\varepsilon.$$

Obviously, z and z'' belong to L_2, and z'' can be written as $z'' = z_0 + z_1 + z_2$, where $z_0 = x''\zeta_\varepsilon$, $z_1 = 2x'\zeta'_\varepsilon$, $z_2 = x\zeta''_\varepsilon$. By (17),

$$z'(0) - (Tz)(0) = -i\nu(h) \int \varphi_0(\tau)\, \{\widetilde{z_0}(\tau) + \widetilde{z_1}(\tau) + \widetilde{z_2}(\tau)\}\, d\tau, \tag{18}$$

with $\varphi_0(\tau) = \varphi(2\pi\nu\tau)$. We will take the limit of this relation as $\varepsilon \to 0$. Obviously, $z'(0) = x'(0)$, and $(Tz)(0) \to (Tx)(0)$ as $\varepsilon \to 0$. Consider the integrals $J_j(\varepsilon) = \int \varphi_0(\tau)\, \widetilde{z}_j(\tau)\, d\tau$ constituting the right-hand side of (18). The function φ_0 belongs to L_2, thus, using the Hölder inequality and the Parseval equality, we obtain

$$|J_1(\varepsilon)| \le \|\varphi_0\|_2\, \|\widetilde{z}_1\|_2 = \|\varphi_0\|_2\, \|z_1\|_2 \le 2\, \|\varphi_0\|_2\, \|x'\|_C\, \|\zeta'_\varepsilon\|_2 = 2\, \|\varphi_0\|_2\, \|x'\|_C\, \|\zeta'\|_2\, \varepsilon^{\frac{1}{2}}.$$

We see that $J_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. In a similar way one can show that $J_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. Now let us investigate the behaviour of $J_0(\varepsilon)$. The Fourier transform of the function $z_0 = x''\zeta_\varepsilon$ is the convolution

$$\widetilde{z}_0(\tau) = \int \widetilde{\zeta}_\varepsilon(\tau - t)\, d\mu_x(t).$$

It follows that

$$J_0(\varepsilon) = \int \varphi_0(\tau)\, \widetilde{z}_0(\tau)\, d\tau = \int \widetilde{\zeta}_\varepsilon(t) \int \varphi_0(t + \tau)\, d\mu(\tau)\, dt.$$

The family of the functions $\widetilde{\zeta}_\varepsilon$ is δ-shaped, consequently, $J_0(\varepsilon)$ tends to $\int \varphi_0(\tau)\, d\mu(\tau)$ as $\varepsilon \to 0$. Thus, the limit of (18) as $\varepsilon \to 0$ is

$$x'(0) - (Tx)(0) = -i\nu \int \varphi_0(\tau)\, d\mu_x(\tau), \quad x \in F_2.$$

This is equivalent to the fact that representation (16) holds for each function $x \in F_2$.

Using (16) and (15), one can estimate quantity (3) for operator (13) from above, namely,

$$u_{1,2}(T_{1,2}) \le \nu(h)\, \|\varphi\|_C = \frac{h}{2}.$$

By (14), this yields

$$e_{1,2}(N_{1,2}(h)) \le \frac{h}{2}. \tag{19}$$

Moreover, inequalities (11) and (19) give the estimate

$$K^2 \le \frac{32}{\pi^3} \sum_{\ell=0}^{\infty} \frac{1}{(2\ell + 1)^3} \tag{20}$$

for the best constant in the inequality

$$\|x'\|_C \le K\, (\|x\|_C\, \|x''\|_V)^{\frac{1}{2}} \tag{21}$$

which is a particular case of (7).

Now we will derive statements converse to (20) and (19). For, consider the function

$$\chi(t) = \chi_{1,2}(t) = \frac{1}{2} \int_0^\pi \frac{\pi - u}{\sin u}\, \sin ut\, du. \tag{22}$$

Obviously, χ is an odd entire function. Furthermore, since

$$\sum_{j=0}^{m-1} \sin{(2j+1)u} = \frac{\sin^2 mu}{\sin u} = \frac{1 - \cos 2mu}{2 \sin u},$$

we have

$$\chi(t) = \frac{1}{2} \int_0^\pi \frac{\pi - u}{\sin u}\, \sin ut\, du = \sum_{j=0}^{m-1} \varphi_j(t) + \frac{1}{2} \int_0^\pi \frac{\pi - u}{\sin u}\, \sin ut\, \cos 2mu\, du, \tag{23}$$

where

$$\varphi_j(t) = \int_0^\pi (\pi - u)\,\sin{(2j+1)u}\,\sin ut\,du.$$

Each of the functions φ_j is entire and it is easy to check that

$$\varphi_j(2j+1) = \frac{\pi^2}{4},$$

$$\varphi_j(t) = \frac{1+\cos t\pi}{2}\left\{\frac{1}{(2j+1-t)^2} - \frac{1}{(2j+1+t)^2}\right\}, \quad t \neq 2j+1. \tag{24}$$

For a fixed t, the value of the last integral in (23) tends to zero as $m \to \infty$, therefore

$$\chi(t) = \sum_{j=0}^{\infty} \varphi_j(t) = \frac{1+\cos t\pi}{2} \sum_{j=0}^{\infty} \left\{\frac{1}{(2j+1-t)^2} - \frac{1}{(2j+1+t)^2}\right\}. \tag{25}$$

It follows from (24) that $\varphi_j(t) \geq 0$ for $t \geq 0$ and $\varphi_j(2m+1) = 0$ for $j \neq m$. Hence, the function χ is non-negative on the half-line $(0,\infty)$, and

$$\chi(2j+1) = \frac{\pi^2}{4}, \quad j = 0,1,\ldots\,. \tag{26}$$

Using the well-known identity

$$\frac{1}{\sin^2 \pi t} = \frac{1}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(t-k)^2},$$

we obtain

$$\frac{\pi^2}{4} = \frac{1+\cos t\pi}{2} \sum_{j=0}^{\infty} \left\{\frac{1}{(2j+1-t)^2} + \frac{1}{(2j+1+t)^2}\right\}. \tag{27}$$

It follows from relations (25)–(27) that

$$0 \leq \chi(t) \leq \frac{\pi^2}{4}, \quad t \geq 0,$$

$$\|\chi\|_{C(-\infty,\infty)} = \chi(2j+1) = \frac{\pi^2}{4}, \quad j \geq 0. \tag{28}$$

Further on, using (25) we find that

$$\chi'(0) = \frac{1}{2}\int_0^\pi \frac{u(\pi-u)}{\sin u}\,du = 4\sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^3}. \tag{29}$$

Now let us calculate the integral

$$J = \frac{1}{2}\int_0^\pi \frac{u^2(\pi-u)}{\sin u}\,du.$$

Taking $\pi - u$ as a new variable, we obtain

$$J = \frac{1}{4}\int_0^\pi \frac{u^2(\pi-u)+(\pi-u)^2 u}{\sin u}\,du = \frac{\pi}{4}\int_0^\pi \frac{u(\pi-u)}{\sin u}\,du = 2\pi\sum_{\ell=0}^{\infty}\frac{1}{(2\ell+1)^3}.$$

Denote by y the odd function which vanishes for $u > \pi$ and is $y(u) = \frac{\pi-u}{4\sin u}$ for $u \in (0,\pi)$. The inverse Fourier transform $z = \hat{y}$ of this function

$$z(t) = \hat{y}(t) = \frac{i}{2}\int_0^\pi \frac{\pi-u}{\sin u}\,\sin 2\pi tu\,du$$

is equal to $i\chi(2\pi t)$. Therefore,

$$\|z\|_C = \frac{\pi^2}{4},$$

$$z'(0) = i2\pi\chi'(0) = 8\pi i \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^3},$$

$$\|\widetilde{z''}\|_1 = (2\pi)^2 \int_0^\pi u^2 y(u)\,du = (2\pi)^3 \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^3}.$$

Thus, the function z belongs to F_2 and provides the following estimate from below for the best constant K in (21):

$$K^2 \geq \frac{|z'(0)|^2}{\|z\|_C\,\|\widetilde{z''}\|_1} = \frac{32}{\pi^3} \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^3}. \tag{30}$$

Inequalities (30), (20), (19), (11) imply the relations

$$K_{1,2}^2 = \frac{32}{\pi^3} \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^3}, \quad e_{1,2}(N_{1,2}(h)) = \frac{h}{2}.$$

This proves Theorems 1 and 2 for $n = 2$, $k = 1$.

In the author's paper [3], the solution of problem (2) for the class

$$\widetilde{Q}_n(S) = \{x \in S : \|\widetilde{x^{(n)}}\|_1 \leq 1\} \subset \widetilde{Q}_n$$

was, in fact, given, and the value of the best constant $K_{k,n}(S)$ in inequality (7) on the set of functions $x \in S$ was determined for $n \geq 3$, $1 \leq k \leq n-1$. One could use these results to prove Theorems 1 and 2 for $n \geq 3$. However, we give here a different proof, or, more exactly, a sketch of the proof.

Now assume that $n \geq 3$, $k = 1$. Let η be a 2π-periodic odd function which is defined on $[0, \pi]$ by the formulae

$$\eta(t) = t - \frac{1}{n}\left(\frac{2}{\pi}\right)^{n-1} t^n, \quad t \in \left[0, \frac{\pi}{2}\right],$$

$$\eta(t) = \eta(\pi - t), \quad t \in \left[\frac{\pi}{2}, \pi\right].$$

Using the function η, we define a function φ on the real line by $\varphi(t) = (t - \eta(t))\,t^{-n}$. The functions η and φ satisfy the following properties (see [3, proof of Theorem 4.1]):

$$\eta(t) = \sum_{\ell=0}^{\infty} c_\ell \sin(2\ell+1)t, \quad (-1)^\ell c_\ell \geq 0, \quad \ell \geq 0,$$

$$\|\eta\|_C = \sum_{\ell=0}^{\infty} |c_\ell| = \eta\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\,\frac{n-1}{n},$$

$$\|\varphi\|_C = \varphi\left(\frac{\pi}{2}\right) = \frac{1}{n}\left(\frac{2}{\pi}\right)^{n-1}.$$

Now take $h > 0$, put $\nu = \nu(h) = \frac{\pi h}{2}$ and define an operator $T = T_{1,n}$ in C by the formula

$$(T_{1,n}x)(t) = \frac{1}{2\nu(h)} \sum_{\ell=0}^{\infty} c_\ell\,\{x(t + (2\ell+1)\nu) - x(t - (2\ell+1)\nu)\}. \tag{31}$$

It is clear that $T_{1,n}$ is a linear bounded operator in C and

$$\|T_{1,n}\|_{C\to C} = \frac{1}{\nu}\sum_{\ell=0}^{\infty}|c_\ell| = \frac{n-1}{nh}. \tag{32}$$

As in the proof for $n=2$ above, one can show that the representation

$$x'(t) - (T_{1,n}x)(t) = (-i\nu)^{n-1}\int e^{2\pi t\tau i}\,\varphi(2\pi\nu\tau)\,d\widetilde{x^{(n)}}(\tau) \tag{33}$$

holds for all functions $x \in F_n$. It follows from this representation that

$$u_{1,n}(T_{1,n}) \le \nu^{n-1}\,\|\varphi\|_C = \frac{h^{n-1}}{n}. \tag{34}$$

In the case $n=3$, $k=2$, denote by η the even 2π-periodic function, defined on $[0,\pi]$ by the formulae

$$\eta(t) = t^2 - \frac{4}{3\pi}\,t^3, \quad t \in \left[0, \frac{\pi}{2}\right],$$

$$\eta(t) = \eta(\pi - t), \quad t \in \left[\frac{\pi}{2}, \pi\right].$$

We have

$$\eta(t) = \sum_{\ell=1}^{\infty} c_\ell\,(1-\cos 2\ell t), \quad c_\ell = \frac{1-(-1)^\ell}{\ell^4}\,\frac{2}{\pi^3}.$$

It follows that

$$2\sum_{\ell=1}^{\infty} c_\ell = 2\sum_{j=0}^{\infty} c_{2j+1} = \|\eta\|_C = \eta\left(\frac{\pi}{2}\right) = \frac{\pi^2}{12}.$$

Moreover, it is easy to see that the function $\varphi(t) = (t^2 - \eta(t))t^{-3}$ satisfies the property

$$\|\varphi\|_C = \varphi\left(\frac{\pi}{2}\right) = \frac{4}{3\pi}.$$

Now we define a bounded linear operator $T_{2,3}$ in the space C by the formula

$$(T_{2,3}x)(t) = -\frac{1}{2\nu^2}\sum_{\ell=1}^{\infty} c_\ell\{x(t+2\ell\nu) - 2x(t) + x(t-2\ell\nu)\}. \tag{35}$$

For this operator we have

$$\|T_{2,3}\|_{C\to C} \le \frac{2}{\nu^2}\sum_{\ell=1}^{\infty}|c_\ell| = \frac{h^{-2}}{3}. \tag{36}$$

For each $x \in F_3$ we have the representation

$$x''(t) - (T_{2,3}x)(t) = -\nu\int e^{2\pi t\tau i}\,\varphi(2\pi\tau\nu)\,d\widetilde{x'''}(\tau); \tag{37}$$

it follows from this representation that

$$u_{2,3}(T_{2,3}) \le \nu\,\|\varphi\|_C = \frac{2h}{3}. \tag{38}$$

Now we define an operator $T_{k,n}$ for arbitrary $n \ge 3$, $1 \le k < n$ by the formula

$$T_{k,n} = T_{m,n-k+m}\,T_{k-m,m}\,, \quad 0 < m < k < n. \tag{39}$$

For example, we can take $m = k-1$, then (39) takes the form

$$T_{k,n} = T_{k-1,n-1}\, T_{1,n}, \tag{40}$$

and if the operator $T_{k,n}$ is defined for all $3 \le n \le \bar{n}$, $1 \le k \le n-1$ (and $n = \bar{n}$, $k = 1$), then using (40) we define $T_{k,n}$ for $n = \bar{n}$, $k = 2, \dots, \bar{n}-1$. Let us check that

$$\|T_{k,n}\| \le \frac{n-k}{n}\, h^{-k}, \tag{41}$$

$$u_{k,n}(T_{k,n}) \le \frac{k}{n}\, h^{n-k} \tag{42}$$

for all $n \ge 3$, $1 \le k < n$. For $k = 1$, $n \ge 3$ and for $k = 2$, $n = 3$ these relations coincide with (38), (36), (34), (32). For the other values of the parameters k, n we have

$$\|T_{k,n}\| \le \|T_{m,n-k+m}\|\, \|T_{k-n,m}\|. \tag{43}$$

For $x \in F_n$, write $x^{(k)} - T_{k,n}x$ in the form

$$x^{(k)} - T_{k,n}x = \frac{d^m\, x^{(k-m)}}{dt^m} - T_{m,n-k+m}x^{(k-m)} + T_{m,n-k+m}(x^{(k-m)} - T_{k-m,n}x).$$

This representation gives the estimate

$$u_{k,n}(T_{k,n}) \le u_{m,k-n+m}(T_{m,n-k+m}) + \|T_{m,n-k+m}\|\, u_{k-m,n}(T_{k-m,n}). \tag{44}$$

Inequalities (41) and (42) follow from estimates (43) and (44) by induction.

Statements (41), (42) imply the following estimate from above for quantity (2) for $n \ge 3$, $1 \le k < n$:

$$e_{k,n}\left(\frac{n-k}{n}\, h^{-k}\right) \le \frac{k}{n}\, h^{n-k}, \quad h > 0. \tag{45}$$

Inequality (11) gives the estimate from above

$$K_{k,n} \le 1 \tag{46}$$

for the best constant in inequality (7).

On the other hand, the function $\psi(t) = \sin t$ belongs to F_n for each n, and

$$\|\psi\|_C = \|\psi^{(k)}\|_C = \|\psi^{(n)}\|_V = 1.$$

This function provides estimates that are converse to (45), (46). Thus, equalities (4), (8) are valid for all $n \ge 3$, $1 \le k < n$. This completes the proofs of Theorems 1, 2. $\square$

Remark. We also have proved that

$$\|T_{k,n}\|_{C\to C} = N_{k,n}(h),$$

$$e_{k,n}(N_{k,n}(h)) = u_{k,n}(T_{k,n}(h)) = \frac{k}{n}\, h^{n-k}$$

for all $n \ge 2$, $1 \le k < n$, i. e., the operators $T_{k,n}$ are extremal operators in problem (2). Moreover, the sine function for $n \ge 3$, $1 \le k < n$ and the function χ defined by (22) for $n = 2$, $k = 1$ are extremal in inequality (7), i. e., inequality (7) turns into an equality for them.

Problem (2) is connected to one further similar problem. Denote by W_n^r the set of functions $x \in L_r \cap L_2$ such that their derivatives $x^{(n-1)}$ are locally absolutely continuous, and $x^{(n)} \in L_2$.

Consider the subclass $Q_n^r = \{x \in W_n^r : \|x^{(n)}\|_2 \le 1\}$ in the set W_n^r. For a linear bounded operator T in L_r consider the quantity

$$U(T) = \sup\{\|x^{(k)} - Tx\|_2 : x \in Q_n^r\}.$$

We are interested in the quantity

$$E_{k,n}(N)_r = \inf\{U(T) : T \in \mathcal{L}_r(N)\} \tag{47}$$

of the best approximation of the differentiation operator of order k in the space L_2 on the class Q_n^r by the set $\mathcal{L}_r(N)$ of linear bounded operators in L_r with the norm $\|T\| = \|T\|_{L_r \to L_r} \le N$; for $r = \infty$ we consider the space C of continuous functions in the place of L_r.

For $r = 2$ and all $n \ge 2$, $1 \le k \le n-1$, problem (47) was solved by Yu. N. Subbotin and L. V. Taikov [9]; in particular, they gave an extremal operator $T_{k,n}^0$ which provides the lower bound in (47). The author's paper [3] gives a solution of problem (47) for $1 \le r \le \infty$ and $n \ge 3$ $(1 \le k < n)$. Namely, it is shown that

$$E_{k,n}\left(\frac{n-k}{n}\, h^{-k}\right)_r = \frac{k}{n}\, h^{n-k}, \quad h > 0, \tag{48}$$

and an extremal operator is the one defined by formulae (31), (35), (40); this operator differs from the operator $T_{k,n}^0$ from [9] and does not depend on r. According to a result from [9] for $r = 2$, formula (48) is also valid for $n = 2$, $k = 1$. In what follows we will show that, in contrast to the case when $n \ge 3$, the quantity $E_{1,2}(N)_r$, in general, depends on r, namely, $E_{1,2}(N)_\infty > E_{1,2}(N)_2$. We will see that $e_{1,2}(N) = E_{1,2}(N)_\infty$ and extremal operators in these problems coincide, so that problem (2) and problem (47) for $r = \infty$ coincide for all $n \ge 2$, $1 \le k \le n-1$. The reason for this behaviour has been explained in the author's papers [2,3]; it is, in particular, connected to the fact that, in (47), it is enough to consider only operators $T \in \mathcal{L}_r(N)$ which are shift-invariant. The following statement holds.

Theorem 3. *If $n = 2$, $k = 1$, $r = \infty$, then for each $h > 0$ we have*

$$E_{1,2}(N_{1,2}(h))_\infty = \frac{h}{2}, \quad N_{1,2}(h) = \frac{16}{\pi^3 h} \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^3}, \tag{49}$$

and the operator $T_{1,2}$ defined in (13) *is extremal.*

P r o o f. Representation (17) holds for functions $x \in W_2^\infty$. Therefore,

$$\|x' - T_{1,2}x\|_2 = \nu\, \|\varphi_0\, \widetilde{x''}\|_2 \le \nu\, \|\varphi\|_C\, \|\widetilde{x''}\|_2 = \nu\, \|\varphi\|_C\, \|x''\|_2,$$

and, consequently, $U(T_{1,2}) \le \frac{h}{2}$. Moreover, $\|T_{1,2}\|_{C \to C} = N_{1,2}(h)$. Hence,

$$E_{1,2}(N_{1,2}(h))_\infty \le \frac{h}{2}. \tag{50}$$

It follows from Theorem 3.1 in [3] that (cf. (11))

$$2(N\, E_{1,2}(N)_\infty)^{\frac{1}{2}} \ge K(S), \tag{51}$$

where $K(S)$ is the best constant in inequality (21) on the set S. Let us prove that $K(S) = K_{1,2}$. Consider the family of the functions $\chi_\varepsilon(t) = e^{-\varepsilon^2 t^2}\, \chi(t)$, where the function χ is defined by (22). It is easy to see that $\chi_\varepsilon \in S$, $\chi_\varepsilon'(0) = \chi'(0)$, and $\|\chi_\varepsilon\|_C \to \|\chi\|_C$, $\|\widetilde{\chi_\varepsilon''}\|_1 \to \|\widetilde{\chi''}\|_1$ as $\varepsilon \to 0$. From these facts we conclude that $K(S) \ge K_{1,2}$, and, consequently, $K(S) = K_{1,2}$. This yields an inequality converse to (50) and thus proves Theorem 3. $\square$

Remark. The operator $T_{1,2}$ is also extremal in problem (47) for $r = 2$, but

$$\|T_{1,2}\|_{L_2 \to L_2} = \frac{1}{2h} < \|T_{1,2}\|_{C \to C}. \tag{52}$$

One can conjecture that the operator $T_{1,2}$ is extremal for all r $(1 \le r \le \infty)$.

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